New results on the least common multiple of consecutive integers

Bakir FARHI and Daniel KANE

Département de Mathématiques, Université du Maine,
Avenue Olivier Messiaen, 72085 Le Mans Cedex 9, France.
bakir.farhi@gmail.com

Harvard University, Department of Mathematics
1 Oxford Street, Cambridge MA 02139, USA.
aladkeenin@gmail.com

Abstract

When studying the least common multiple of some finite sequences of integers, the first author introduced the interesting arithmetic functions g_k ($k \in \mathbb{N}$), defined by $g_k(n) := \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}(n,n+1,\dots,n+k)}$ ($\forall n \in \mathbb{N} \setminus \{0\}$). He proved that g_k ($k \in \mathbb{N}$) is periodic and $k!$ is a period of g_k. He raised the open problem consisting to determine the smallest positive period P_k of g_k. Very recently, S. Hong and Y. Yang have improved the period $k!$ of g_k to $\mathrm{lcm}(1, 2, \dots, k)$. In addition, they have conjectured that P_k is always a multiple of the positive integer $\frac{\mathrm{lcm}(1,2,\dots,k,k+1)}{k+1}$. An immediate consequence of this conjecture states that if $(k+1)$ is prime then the exact period of g_k is precisely equal to $\mathrm{lcm}(1, 2, \dots, k)$.

In this paper, we first prove the conjecture of S. Hong and Y. Yang and then we give the exact value of P_k ($k \in \mathbb{N}$). We deduce, as a corollary, that P_k is equal to the part of $\mathrm{lcm}(1, 2, \dots, k)$ not divisible by some prime.

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1 Introduction

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N} \setminus \{0\}$ of positive integers.

Many results concerning the least common multiple of sequences of integers are known. The most famous is nothing else than an equivalent of the prime number theorem; it sates that $\log \mathrm{lcm}(1, 2, \dots, n) \sim n$ as n tends to infinity (see e.g., [6]). Effective bounds for $\mathrm{lcm}(1, 2, \dots, n)$ are also given by several authors (see e.g., [5] and [10]).

Recently, the topic has undergone important developments. In [1], Bateman, Kalb and Stenger have obtained an equivalent for $\log \mathrm{lcm}(u_1, u_2, \dots, u_n)$ when $(u_n)_n$ is an arithmetic progression. In [2], Cilleruelo has obtained a simple equivalent for the least common multiple of a quadratic progression. For the effective bounds, Farhi [3] [4] got lower bounds for $\mathrm{lcm}(u_0, u_1, \dots, u_n)$ in both cases when $(u_n)_n$ is an arithmetic progression or when it is a quadratic progression. In the case of arithmetic progressions, Hong and Feng [7] and Hong and Yang [8] obtained some improvements of Farhi's lower bounds.

Among the arithmetic progressions, the sequences of consecutive integers are the most well-known with regards the properties of their least common multiple. In [4], Farhi introduced the arithmetic function $g_k : \mathbb{N}^* \to \mathbb{N}^*$ $(k \in \mathbb{N})$ which is defined by:

$$g_k(n) := \frac{n(n+1)\dots(n+k)}{\mathrm{lcm}(n, n+1, \dots, n+k)} \qquad (\forall n \in \mathbb{N}^*).$$

Farhi proved that the sequence $(g_k)_{k\in\mathbb{N}}$ satisfies the recursive relation:

$$g_k(n) = \gcd\left(k!, (n+k)g_{k-1}(n)\right) \qquad (\forall k, n \in \mathbb{N}^*). \tag{1}$$

Then, using this relation, he deduced (by induction on k) that g_k $(k \in \mathbb{N})$ is periodic and $k!$ is a period of g_k. A natural open problem raised in [4] consists to determine the exact period (i.e., the smallest positive period) of g_k.

For the following, let P_k denote the exact period of g_k. So, Farhi's result amounts that P_k divides $k!$ for all $k \in \mathbb{N}$. Very recently, Hong and Yang have shown that P_k divides $\mathrm{lcm}(1, 2, \dots, k)$. This improves Farhi's result but it doesn't solve the raised problem of determining the P_k's. In their paper [8], Hong and Yang have also conjectured that P_k is a multiple of $\frac{\mathrm{lcm}(1,2,\dots,k+1)}{k+1}$ for all nonnegative integer k. According to the property that P_k divides $\mathrm{lcm}(1, 2, \dots, k)$ $(\forall k \in \mathbb{N})$, this conjecture implies that the equality $P_k = \mathrm{lcm}(1, 2, \dots, k)$ holds at least when $(k + 1)$ is prime.

In this paper, we first prove the conjecture of Hong and Yang and then we give the exact value of P_k $(\forall k \in \mathbb{N})$. As a corollary, we show that P_k is equal to the part of $\mathrm{lcm}(1, 2, \dots, k)$ not divisible by some prime and that the equality $P_k = \mathrm{lcm}(1, 2, \dots, k)$ holds for an infinitely many $k \in \mathbb{N}$ for which $(k + 1)$ is not prime.

2 Proof of the conjecture of Hong and Yang

We begin by extending the functions g_k $(k \in \mathbb{N})$ to $\mathbb{Z}$ as follows:
- We define $g_0 : \mathbb{Z} \to \mathbb{N}^*$ by $g_0(n) = 1$, $\forall n \in \mathbb{Z}$.
- If, for some $k \geq 1$, g_{k-1} is defined, then we define g_k by the relation:

$$g_k(n) = \gcd\left(k!, (n+k)g_{k-1}(n)\right) \qquad (\forall n \in \mathbb{Z}). \tag{1'}$$

Those extensions are easily seen to be periodic and to have the same period as their restriction to $\mathbb{N}^*$. The following proposition plays a vital role in what follows:

Proposition 2.1 *For any $k \in \mathbb{N}$, we have $g_k(0) = k!$.*

Proof. This follows by induction on k with using the relation $(1')$. ■

We now arrive at the theorem implying the conjecture of Hong and Yang.

Theorem 2.2 *For all $k \in \mathbb{N}$, we have:*

$$P_k = \frac{\mathrm{lcm}(1, 2, \ldots, k+1)}{k+1} . \gcd\left(P_k + k + 1, \mathrm{lcm}(P_k + 1, P_k + 2, \ldots, P_k + k)\right).$$

The proof of this theorem needs the following lemma:

Lemma 2.3 *For all $k \in \mathbb{N}$, we have:*

$$\mathrm{lcm}(P_k, P_k + 1, \ldots, P_k + k) = \mathrm{lcm}(P_k + 1, P_k + 2, \ldots, P_k + k).$$

Proof of the Lemma. Let $k \in \mathbb{N}$ fixed. The required equality of the lemma is clearly equivalent to say that P_k divides $\mathrm{lcm}(P_k + 1, P_k + 2, \ldots, P_k + k)$. This amounts to showing that for any prime number p:

$$v_p(P_k) \leq v_p\left(\mathrm{lcm}(P_k + 1, \ldots, P_k + k)\right) = \max_{1 \leq i \leq k} v_p(P_k + i). \tag{2}$$

So it remains to show (2). Let p be a prime number. Because P_k divides $\mathrm{lcm}(1, 2, \ldots, k)$ (according to the result of Hong and Yang [8]), we have $v_p(P_k) \leq v_p(\mathrm{lcm}(1, 2, \ldots, k))$, that is $v_p(P_k) \leq \max_{1 \leq i \leq k} v_p(i)$. So there exists $i_0 \in \{1, 2, \ldots, k\}$ such that $v_p(P_k) \leq v_p(i_0)$. It follows, according to the elementary properties of the p-adic valuation, that we have:

$$v_p(P_k) = \min\left(v_p(P_k), v_p(i_0)\right) \leq v_p(P_k + i_0) \leq \max_{1 \leq i \leq k} v_p(P_k + i),$$

which confirms (2) and completes this proof. ■

Proof of Theorem 2.2. Let $k \in \mathbb{N}$ fixed. The main idea of the proof is to calculate in two different ways the quotient $\frac{g_k(P_k)}{g_k(P_k+1)}$ and then to compare the obtained results. On one hand, we have from the definition of the function g_k:

$$\begin{aligned}
\frac{g_k(P_k)}{g_k(P_k + 1)} &= \frac{P_k(P_k + 1) \ldots (P_k + k)}{\mathrm{lcm}(P_k, P_k + 1, \ldots, P_k + k)} \Big/ \frac{(P_k + 1)(P_k + 2) \ldots (P_k + k + 1)}{\mathrm{lcm}(P_k + 1, P_k + 2, \ldots, P_k + k + 1)} \\
&= P_k \frac{\mathrm{lcm}(P_k + 1, P_k + 2, \ldots, P_k + k + 1)}{(P_k + k + 1)\mathrm{lcm}(P_k, P_k + 1, \ldots, P_k + k)} \qquad (3)
\end{aligned}$$

Next, using Lemma 2.3 and the well-known formula "$ab = \mathrm{lcm}(a, b)\gcd(a, b)$ $(\forall a, b \in \mathbb{N}^*)$", we have:

$$(P_k + k + 1)\mathrm{lcm}(P_k, P_k + 1, \ldots, P_k + k) = (P_k + k + 1)\mathrm{lcm}(P_k + 1, P_k + 2, \ldots, P_k + k)$$

$$= \operatorname{lcm}\left(P_k+k+1, \operatorname{lcm}(P_k+1,\dots,P_k+k)\right)$$
$$\times \gcd\left(P_k+k+1, \operatorname{lcm}(P_k+1,\dots,P_k+k)\right)$$
$$= \operatorname{lcm}(P_k+1,P_k+2,\dots,P_k+k+1)\gcd\left(P_k+k+1, \operatorname{lcm}(P_k+1,\dots,P_k+k)\right).$$

By substituting this into (3), we obtain:

$$\frac{g_k(P_k)}{g_k(P_k+1)} = \frac{P_k}{\gcd\left(P_k+k+1, \operatorname{lcm}(P_k+1,\dots,P_k+k)\right)}. \tag{4}$$

On other hand, according to Proposition 2.1 and to the definition of P_k, we have:

$$\frac{g_k(P_k)}{g_k(P_k+1)} = \frac{k!}{g_k(1)} = \frac{\operatorname{lcm}(1,2,\dots,k+1)}{k+1}. \tag{5}$$

Finally, by comparing (4) and (5), we get:

$$P_k = \frac{\operatorname{lcm}(1,2,\dots,k+1)}{k+1}\gcd\left(P_k+k+1, \operatorname{lcm}(P_k+1,P_k+2,\dots,P_k+k)\right),$$

as required. The proof is complete. ∎

From Theorem 2.2, we derive the following interesting corollary, which confirms the conjecture of Hong and Yang [8].

Corollary 2.4 *For all $k \in \mathbb{N}$, the exact period P_k of g_k is a multiple of the positive integer $\frac{\operatorname{lcm}(1,2,\dots,k,k+1)}{k+1}$. In addition, for all $k \in \mathbb{N}$ for which $(k+1)$ is prime, we have precisely $P_k = \operatorname{lcm}(1,2,\dots,k)$.*

Proof. The first part of the corollary immediately follows from Theorem 2.2. Furthermore, we remark that if k is a natural number such that $(k+1)$ is prime, then we have $\frac{\operatorname{lcm}(1,2,\dots,k+1)}{k+1} = \operatorname{lcm}(1,2,\dots,k)$. So, P_k is both a multiple and a divisor of $\operatorname{lcm}(1,2,\dots,k)$. Hence $P_k = \operatorname{lcm}(1,2,\dots,k)$. This finishes the proof of the corollary. ∎

Now, we exploit the identity of Theorem 2.2 in order to obtain the p-adic valuation of P_k ($k \in \mathbb{N}$) for most prime numbers p.

Theorem 2.5 *Let $k \geq 2$ be an integer and $p \in [1,k]$ be a prime number satisfying:*

$$v_p(k+1) < \max_{1 \leq i \leq k} v_p(i). \tag{6}$$

Then, we have:

$$v_p(P_k) = \max_{1 \leq i \leq k} v_p(i).$$

Proof. The identity of Theorem 2.2 implies the following equality:

$$v_p(P_k) = \max_{1 \leq i \leq k+1}(v_p(i)) - v_p(k+1) + \min\left\{v_p(P_k+k+1), \max_{1 \leq i \leq k}(v_p(P_k+i))\right\}. \tag{7}$$

Now, using the hypothesis (6) of the theorem, we have:

$$\max_{1\le i\le k+1}(v_p(i)) = \max_{1\le i\le k}(v_p(i)) \tag{8}$$

and

$$\max_{1\le i\le k+1}(v_p(i)) - v_p(k+1) > 0.$$

According to (7), this last inequality implies that:

$$\min\left\{v_p(P_k+k+1), \max_{1\le i\le k} v_p(P_k+i)\right\} < v_p(P_k). \tag{9}$$

Let $i_0 \in \{1,2,\dots,k\}$ such that $\max_{1\le i\le k} v_p(i) = v_p(i_0)$. Since P_k divides $\mathrm{lcm}(1,2,\dots,k)$, we have $v_p(P_k) \le v_p(i_0)$, which implies that $v_p(P_k+i_0) \ge \min(v_p(P_k), v_p(i_0)) = v_p(P_k)$. Thus $\max_{1\le i\le k} v_p(P_k+i) \ge v_p(P_k)$. It follows from (9) that

$$\min\left\{v_p(P_k+k+1), \max_{1\le i\le k} v_p(P_k+i)\right\} = v_p(P_k+k+1) < v_p(P_k). \tag{10}$$

So, we have

$$\min\left(v_p(P_k), v_p(k+1)\right) \le v_p(P_k+k+1) < v_p(P_k),$$

which implies that

$$v_p(k+1) < v_p(P_k)$$

and then, that

$$v_p(P_k+k+1) = \min\left(v_p(P_k), v_p(k+1)\right) = v_p(k+1).$$

According to (10), it follows that

$$\min\left\{v_p(P_k+k+1), \max_{1\le i\le k} v_p(P_k+i)\right\} = v_p(k+1). \tag{11}$$

By substituting (8) and (11) into (7), we finally get:

$$v_p(P_k) = \max_{1\le i\le k} v_p(i),$$

as required. The theorem is proved. ■

Using Theorem 2.5, we can find infinitely many natural numbers k so that $(k+1)$ is not prime and the equality $P_k = \mathrm{lcm}(1,2,\dots,k)$ holds. The following corollary gives concrete examples for such numbers k.

Corollary 2.6 *If k is an integer having the form $k = 6^r - 1$ ($r \in \mathbb{N}, r \ge 2$), then we have*

$$P_k = \mathrm{lcm}(1,2,\dots,k).$$

Consequently, there are an infinitely many $k \in \mathbb{N}$ for which $(k+1)$ is not prime and the equality $P_k = \mathrm{lcm}(1,2,\dots,k)$ holds.

Proof. Let $r \geq 2$ be an integer and $k = 6^r - 1$. We have $v_2(k+1) = v_2(6^r) = r$ while $\max_{1 \leq i \leq k} v_2(i) \geq r+1$ (since $k \geq 2^{r+1}$). Thus $v_2(k+1) < \max_{1 \leq i \leq k} v_2(i)$. Similarly, we have $v_3(k+1) = v_3(6^r) = r$ while $\max_{1 \leq i \leq k} v_3(i) \geq r+1$ (since $k \geq 3^{r+1}$). Thus $v_3(k+1) < \max_{1 \leq i \leq k} v_3(i)$.
Finally, for any prime $p \in [5, k]$, we clearly have $v_p(k+1) = v_p(6^r) = 0$ and $\max_{1 \leq i \leq k} v_p(i) \geq 1$. Hence $v_p(k+1) < \max_{1 \leq i \leq k} v_p(i)$.
This shows that the hypothesis of Theorem 2.5 is satisfied for any prime number p. Consequently, we have for any prime p: $v_p(P_k) = \max_{1 \leq i \leq k} v_p(i) = v_p(\mathrm{lcm}(1, 2, \ldots, k))$. Hence $P_k = \mathrm{lcm}(1, 2, \ldots, k)$, as required. ■

3 Determination of the exact value of P_k

Notice that Theorem 2.5 successfully computes the value of $v_p(P_k)$ for almost all primes p (in fact we will prove in Proposition 3.3 that Theorem 2.5 fails to provide this value for at most one prime). In order to evaluate P_k, all we have left to do is compute $v_p(P_k)$ for primes p so that $v_p(k+1) \geq \max_{1 \leq i \leq k} v_p(i)$. In particular we will prove:

Lemma 3.1 *Let $k \in \mathbb{N}$. If $v_p(k+1) \geq \max_{1 \leq i \leq k} v_p(i)$, then $v_p(P_k) = 0$.*

From which the following result is immediate:

Theorem 3.2 *We have for all $k \in \mathbb{N}$:*

$$P_k = \prod_{p \text{ prime},\ p \leq k} p^{\begin{cases} 0 & \text{if } v_p(k+1) \geq \max_{1 \leq i \leq k} v_p(i) \\ \max_{1 \leq i \leq k} v_p(i) & \text{else} \end{cases}}.$$

In order to prove this result, we will need to look into some of the more detailed divisibility properties of $g_k(n)$. In this spirit we make the following definitions:

Let $S_{n,k} = \{n, n+1, n+2, \ldots, n+k\}$ be the set of integers in the range $[n, n+k]$.

For a prime number p, let $g_{p,k}(n) := v_p(g_k(n))$. Let $P_{p,k}$ be the exact period of $g_{p,k}$. Since a positive integer is uniquely determined by the number of times each prime divides it, $P_k = \mathrm{lcm}_{p \text{ prime}}(P_{p,k})$.

Now note that

$$\begin{aligned} g_{p,k}(n) &= \sum_{m \in S_{n,k}} v_p(m) - \max_{m \in S_{n,k}} v_p(m) \\ &= \sum_{e>0, m \in S_{n,k}} (1 \text{ if } p^e | m) - \sum_{e>0} (1 \text{ if } p^e \text{ divides some } m \in S_{n,k}) \\ &= \sum_{e>0} \max(0, \#\{m \in S_{n,k} : p^e | m\} - 1). \end{aligned}$$

Let $e_{p,k} = \lfloor \log_p(k) \rfloor = \max_{1\le i\le k} v_p(i)$ be the largest exponent of a power of p that is at most k. Clearly there is at most one element of $S_{n,k}$ divisible by p^e if $e > e_{p,k}$, therefore terms in the above sum with $e > e_{p,k}$ are all 0. Furthermore, for each $e \le e_{p,k}$, at least one element of $S_{p,k}$ is divisible by p^e. Hence we have that

$$g_{p,k}(n) = \sum_{e=1}^{e_{p,k}} \left(\#\{m \in S_{n,k} : p^e | m\} - 1\right). \tag{12}$$

Note that each term on the right hand side of (12) is periodic in n with period $p^{e_{p,k}}$ since the condition $p^e|(n+m)$ for fixed m is periodic with period p^e. Therefore $P_{p,k}|p^{e_{p,k}}$. Note that this implies that the $P_{p,k}$ for different p are relatively prime, and hence we have that

$$P_k = \prod_{p \text{ prime},\ p\le k} P_{p,k}.$$

We are now prepared to prove our main result

Proof of Lemma 3.1. Suppose that $v_p(k+1) \ge e_{p,k}$. It clearly suffices to show that $v_p(P_{q,k}) = 0$ for each prime q. For $q \ne p$ this follows immediately from the result that $P_{q,k}|q^{e_{q,k}}$. Now we consider the case $q = p$.

For each $e \in \{1, \ldots, e_{p,k}\}$, since $p^e|k+1$, it is clear that $\#\{m \in S_{n,k} : p^e|m\} = \frac{k+1}{p^e}$, which implies (according to (12)) that $g_{k,n}$ is independent of n. Consequently, we have $P_{p,k} = 1$, and hence $v_p(P_{p,k}) = 0$, thus completing our proof. ■

Note that a slightly more complicated argument allows one to use this technique to provide an alternate proof of Theorem 2.5.

We can also show that the result in Theorem 3.2 says that P_k is basically $\operatorname{lcm}(1, 2, \ldots, k)$.

Proposition 3.3 *There is at most one prime p so that $v_p(k+1) \ge e_{p,k}$. In particular, by Theorem 3.2, P_k is either $\operatorname{lcm}(1, 2, \ldots, k)$, or $\frac{\operatorname{lcm}(1,2,\ldots,k)}{p^{e_{p,k}}}$ for some prime p.*

Proof. Suppose that for two distinct primes, $p, q \le k$ that $v_p(k+1) \ge e_{p,k}$, and $v_q(k+1) \ge e_{q,k}$. Then

$$k+1 \ge p^{v_p(k+1)} q^{v_q(k+1)} \ge p^{e_{p,k}} q^{e_{q,k}} > \min\left(p^{e_{p,k}}, q^{e_{q,k}}\right)^2 = \min\left(p^{2e_{p,k}}, q^{2e_{q,k}}\right).$$

But this would imply that either $k \ge p^{2e_{p,k}}$ or that $k \ge q^{2e_{q,k}}$ thus violating the definition of either $e_{p,k}$ or $e_{q,k}$. ■

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